Tutorial 9

In the following problems, V denotes a finite-dimensional vector space.

1. Suppose $T \in \mathcal{L}(V)$ is such that the characteristic polynomial q and minimal polynomial p of T are

$$q(z) = (z-1)^4(z-5)^4, \qquad\qquad p(z) = (z-1)^2(z-5)^2$$

   If $\dim E(1,T) = 2$ and $\dim E(5,T) = 3$ then what is the Jordan form for T (up to reordering)?

2. Let $T \in \mathcal{L}(V)$ be a linear operator and $\beta \subseteq V$ be a basis for V such that

$${}_\beta[T]_\beta = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \ddots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \lambda & 0 \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

   for some $\lambda \in \mathbb{C}$. What is the Jordan form for T (up to reordering)?

3. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that A is similar to its transpose $A^\top$.

4. The *lower Pascal matrix of size* n is the $n \times n$ matrix P_n defined such that the (i,j)-entry of P_n is $\binom{i-1}{j-1}$. For example,

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

   What is the Jordan form for P_n (up to reordering)?

5. Recall that for a field $\mathbb{F}$, the set of polynomials with coefficients in $\mathbb{F}$ is denoted $\mathbb{F}[x]$. Also recall that for $f \in \mathbb{F}[x]$, the *ideal generated by* f is the set

$$(f) = \{fg \in \mathbb{F}[x] \colon g \in \mathbb{F}[x]\}$$

   This ideal defines an equivalence relation on $\mathbb{F}[x]$. Indeed, we say $h_1, h_2 \in \mathbb{F}[x]$ are equivalent if $h_1 - h_2 \in (f)$. The set of all such equivalence classes is $\mathbb{F}[x]/(f)$.

(a) What familiar vector space is isomorphic to $\mathbb{F}[x]/(x^n)$?

(b) What familiar set does $\mathbb{R}[x]/(x^2+1)$ "behave like"?

6. Suppose V is a real inner product space and let $T \in \mathcal{L}(V_{\mathbb{C}})$. Must there exist $S \in \mathcal{L}(V)$ such that $T = S_{\mathbb{C}}$?

7. Suppose V is a real inner product space. Is it possible to equip $V_{\mathbb{C}}$ with an inner product which restricts to the inner product on V?